GENERALIZED SASAKIAN-SPACE-FORMS WITH D-CONFORMAL CURVATURE TENSOR

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ABSTRACT
In this paper we study generalized Sasakian-space-forms with D-conformal curvature tensor. In generalized Sasakian-space-forms, we investigate some results on D-conformally flat, $\xi$-D-conformally flat, $\phi$-D-conformally flat and the curvature condition $B(\xi, X).S = 0$.

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1. INTRODUCTION

A Sasakian manifold $M(\phi, \xi, \eta, g)$ is said to be a Sasakian-space-form if all the $\phi$-sectional curvatures $K(X \wedge \phi X)$ are equal to a constant $c$, where $K(X \wedge \phi X)$ denotes the sectional curvature of the section spanned by the unit vector field $X$, orthogonal to $\xi$ and $\phi X$. In such a case, the Riemannian curvature tensor of $M$ is given by

$$R(X,Y)Z = \frac{c+3}{4} \{g(Y,Z)X - g(X,Z)Y\}$$
$$+ \frac{c-1}{4} \{g(X,\phi Z)\phi Y - g(Y,\phi Z)\phi X + 2g(X,\phi Y)\phi Z\}$$
$$+ \frac{c-1}{4} \{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X,Z)\eta(Y)\xi$$
$$- g(Y,Z)\eta(X)\xi\}.$$ (1.1)

These spaces can be modeled, depending on $c > -3, c = -3$ or $c < -3$.

As a natural generalization of these manifolds Alegre, Blair and Carriazo introduced and studied the notion of generalized Sasakian-space-forms in 2004 [1]. They replaced constant quantities $(c+3)/4$ and $(c-1)/4$ of relation (1.1) by differentiable functions $f_1, f_2$ and $f_3$.

An almost contact metric manifold $M(\phi, \xi, \eta, g)$ is said to be a generalized Sasakian-space-form if the curvature tensor $R$ is given by [1]
\[ R(X,Y)Z = f_1(g(Y,Z)X - g(X,Z)Y) + f_2(g(X,\varphi Z)\varphi Y - g(Y,\varphi Z)\varphi X + 2g(X,\varphi Y)\varphi Z) \]
\[ + f_3(\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi) \]

where \( f_1, f_2, f_3 \) are differentiable functions on \( M \) and \( X, Y, Z \) are vector fields on \( M \). In such a case manifold is denoted by \( M(f_1, f_2, f_3) \). In [2] authors studied contact metric and trans-Sasakian generalized Sasakian-space-forms. In [5] and [6] authors studied on the locally \( \phi \)-symmetric and \( \eta \)-recurrent Ricci tensor and on the projective curvature tensor respectively. Conformally flat generalized Sasakian-space-forms and locally symmetric generalized Sasakian-space-forms were studied by Kim [7]. Shukla and Shah studied on generalized Sasakian-space-forms with concircular curvature tensor [8].

A Riemannian manifold \( M \) is said to be semi-symmetric if its curvature tensor \( R \) satisfies [9]

\[ R(X,Y)R = 0, \quad X,Y \in TM, \]

where \( R(X,Y) \) acts on \( R \) as a derivation.

Generalized Sasakian-space-forms have also been studied by [10] and others.

2. PRELIMINARIES

In an almost contact metric manifold \( M^{2n+1}(\varphi, \xi, \eta, g) \), where \( \varphi \) is a (1, 1) tensor field, \( \xi \) is a contravariant vector field, \( \eta \) is a 1-form and \( g \) is a compatible Riemannian metric, we have [3]

\[ \varphi^2(X) = -X + \eta(X)\xi, \quad \eta(\xi) = 1, \quad \varphi\xi = 0, \quad \eta(\varphi X) = 0. \]

\[ g(X,\xi) = \eta(X), \]

\[ g(\varphi X,\varphi Y) = g(X,Y) - \eta(X)\eta(Y), \]

\[ g(\varphi X,Y) = -g(X,\varphi Y), \]

\[ (\nabla_X \eta)(Y) = g(\nabla_X \xi, Y). \]

In a \((2n+1)\)-dimensional generalized Sasakian-space-form the following relations hold:

\[ R(X,Y)\xi = (f_1 - f_3}\{\eta(Y)X - \eta(X)Y\}, \]

\[ R(\xi,X)Y = (f_1 - f_3}\{g(X,Y)\xi - \eta(Y)X\}, \]

\[ R(\xi,X)\xi = (f_1 - f_3}\{\eta(X)\xi - X\}. \]
(2.9) \[ S(X,Y) = (2nf_1 + 3f_2 - f_3)g(X,Y) - (3f_2 + (2n-1)f_3)\eta(X)\eta(Y), \]

(2.10) \[ r = 2n(2n+1)f_1 + 6nf_2 - 4nf_3, \]

(2.11) \[ S(X,\xi) = 2n(f_1 - f_3)\eta(X), \]

(2.12) \[ \eta(R(X,Y)Z) = (f_1 - f_3)\{g(Y,Z)\eta(X) - g(X,Z)\eta(Y)\}, \]

(2.13) \[ \eta(R(X,Y)\xi) = 0, \]

(2.14) \[ \eta(R(\xi,X)Y) = (f_1 - f_3)\{g(X,Y) - \eta(X)\eta(Y)\}, \]

(2.15) \[ S(\varphi X, \varphi Y) = S(X,Y) - 2n(f_1 - f_3)\eta(X)\eta(Y). \]

The D-conformal curvature tensor on a Riemannian manifold \((M^{2n+1}, g)\) is defined as [4]

\[
B(X,Y)Z = R(X,Y)Z + \frac{1}{2(n-1)}[S(X,Z)Y - S(Y,Z)X + g(X,Z)QY - g(Y,Z)QX - S(X,Z)\eta(Y)\xi + S(Y,Z)\eta(X)\xi - \eta(X)\eta(Z)QY + \eta(Y)\eta(Z)QX]
\]

(2.16) \[ -\frac{k-2}{2(n-1)}[g(X,Z)Y - g(Y,Z)X]
+ \frac{k}{2(n-1)}[g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi + \eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X] \]

where \(k = \frac{r+4n}{2n-1}\), \(R\) is the curvature tensor, \(S\) is the Ricci tensor and \(r\) is the scalar curvature.

3. RESULTS AND DISCUSSION

Definition: A \((2n+1)\)-dimensional generalized Sasakian-space-form \(M(f_1, f_2, f_3)\) is said to be D-conformally flat if

(3.1) \[ B(X,Y)Z = 0. \]

Theorem 3.1. If a \((2n+1)\)-dimensional generalized Sasakian-space-form \(M(f_1, f_2, f_3)\) is D-conformally flat, then \(f_3 = f_1 + 1\).

Proof. Let us consider a \((2n+1)\)-dimensional generalized Sasakian-space-form which satisfies the condition \(B(X,Y)Z = 0\), then from (2.16) we have
\[
0 = R(X,Y)Z + \frac{1}{2(n-1)}[S(X,Z)Y - S(Y,Z)X + g(X,Z)QY
\]

\[
- g(Y,Z)QX - S(X,Z)\eta(Y)\xi + S(Y,Z)\eta(X)\xi
\]

\[
- \eta(X)\eta(Z)QY + \eta(Y)\eta(Z)QX]
\]

(3.2)

\[
- \frac{k-2}{2(n-1)}[g(X,Z)Y - g(Y,Z)X]
\]

\[
+ \frac{k}{2(n-1)}[g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi]
\]

\[
+ \eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X].
\]

Taking inner product on both sides of (3.2) by \( W \), we get

\[
0 = R(X,Y,Z,W) + \frac{1}{2(n-1)}[S(X,Z)g(Y,W) - S(Y,Z)g(X,W)
\]

\[
+ S(Y,W)g(X,Z) - S(X,W)g(Y,Z) - S(X,Z)\eta(Y)\eta(W)
\]

\[
+ S(Y,Z)\eta(X)\eta(W) - S(Y,W)\eta(X)\eta(Z) + S(X,W)\eta(Y)\eta(Z)]
\]

(3.3)

\[
- \frac{k-2}{2(n-1)}[g(X,Z)g(Y,W) - g(Y,Z)g(X,W)]
\]

\[
+ \frac{k}{2(n-1)}[g(X,Z)\eta(Y)\eta(W) - g(Y,Z)\eta(X)\eta(W)
\]

\[
+ g(Y,W)\eta(X)\eta(Z) - g(X,W)\eta(Y)\eta(Z)].
\]

where \( R(X,Y,Z,W) = g(R(X,Y)Z,W) \).

Setting \( W = \xi \) in (3.3) and using (2.1) and (2.2), we obtain

\[
0 = \eta(R(X,Y)Z) + \frac{1}{2(n-1)}[S(Y,\xi)g(X,Z)
\]

(3.4)

\[
- S(X,\xi)g(Y,Z) - S(Y,\xi)\eta(X)\eta(Z) + S(X,\xi)\eta(Y)\eta(Z)
\]

\[
+ 2\{g(X,Z)\eta(Y) - g(Y,\xi)\eta(X)\}].
\]

Using (2.11) and (2.12) in (3.4), we get

\[
\left(\frac{f_3 - f_1 - 1}{n-1}\right)[g(Y,Z)\eta(X) - g(X,Z)\eta(Y)] = 0.
\]

(3.5)

Since \( g(Y,Z)\eta(X) - g(X,Z)\eta(Y) \neq 0 \), we must have \( f_3 - f_1 - 1 = 0 \), this implies that

(3.6)

\( f_3 = f_1 + 1 \).

This completes the proof of the theorem.
Definition. Generalized Sasakian-space-form $M(f_1, f_2, f_3)$ of dimension $(2n+1)$ is said to be $\xi$-D-conformally flat if

$$ (3.7) \quad B(X,Y)\xi = 0. $$

**Theorem 3.2.** If a $(2n+1)$-dimensional generalized Sasakian-space-form $M(f_1, f_2, f_3)$ satisfies the condition $B(X,Y)\xi = 0$, then $f_3 = f_1 + 1$.

**Proof.** Suppose the condition $B(X,Y)\xi = 0$ holds in a $(2n+1)$-dimensional generalized Sasakian-space-form. Then using (2.1) and (2.2) in (2.16), we have

$$ (3.8) \quad 0 = R(X,Y)\xi + \frac{1}{2(n-1)}[S(X,\xi)Y - S(Y,\xi)X - S(X,\xi)\eta(Y)\xi + S(Y,\xi)\eta(X)\xi] + 2[\eta(X)Y - \eta(Y)X]. $$

In view of (2.6) and (2.11), (3.8) reduces to

$$ (3.9) \quad \left(\frac{f_3 - f_1 - 1}{n-1}\right)[\eta(Y)X - \eta(X)Y] = 0. $$

Since $\eta(Y)X - \eta(X)Y \neq 0$, we must have $f_3 - f_1 - 1 = 0$, this implies that

$$ (3.10) \quad f_3 = f_1 + 1. $$

Hence the theorem is proved.

From theorem 3.1 and theorem 3.2 we obtain the following:

**Corollary 3.1.** In a $(2n+1)$-dimensional generalized Sasakian-space-form $M(f_1, f_2, f_3)$ the curvature conditions $B(X,Y)Z = 0$ and $B(X,Y)\xi = 0$ are equivalent.

**Definition.** Let $M(f_1, f_2, f_3)$ be a $(2n+1)$-dimensional generalized Sasakian-space-form. Then $M(f_1, f_2, f_3)$ is said to be $\varphi$-D-conformally flat if

$$ (3.11) \quad g(B(\varphi X,\varphi Y)\varphi Z,\varphi W) = 0. $$

**Theorem 3.3.** If a $(2n+1)$-dimensional generalized Sasakian-space-form is $\varphi$-D-conformally flat, then it is an $\eta$-Einstein manifold under the condition $Tr.\varphi = 0$.

**Proof.** Let $M(f_1, f_2, f_3)$ be a $(2n+1)$-dimensional generalized Sasakian-space-form. Suppose $M(f_1, f_2, f_3)$ satisfies the condition $g(B(\varphi X,\varphi Y)\varphi Z,\varphi W) = 0$, then from (2.1) and (2.16), we have
\[ g \left( B(\varphi X, \varphi Y) \varphi Z, \varphi W \right) = g \left( R(\varphi X, \varphi Y) \varphi Z, \varphi W \right) \\
+ \frac{1}{2(n-1)} \left[ S(\varphi X, \varphi Z) g(\varphi Y, \varphi W) - S(\varphi Y, \varphi Z) g(\varphi X, \varphi W) \right] \\
+ S(\varphi Y, \varphi W) g(\varphi X, \varphi Z) - S(\varphi X, \varphi W) g(\varphi Y, \varphi Z) \\
- \frac{k-2}{2(n-1)} \left[ g(\varphi X, \varphi Z) g(\varphi Y, \varphi W) - g(\varphi Y, \varphi Z) g(\varphi X, \varphi W) \right]. \]

In view of (3.11) and (3.12), we get

\[ 0 = g \left( R(\varphi X, \varphi Y) \varphi Z, \varphi W \right) + \frac{1}{2(n-1)} \left[ S(\varphi X, \varphi Z) g(\varphi Y, \varphi W) \right] \\
- S(\varphi Y, \varphi Z) g(\varphi X, \varphi W) + S(\varphi Y, \varphi W) g(\varphi X, \varphi Z) \\
- S(\varphi X, \varphi W) g(\varphi Y, \varphi Z) \\
- \frac{k-2}{2(n-1)} \left[ g(\varphi X, \varphi Z) g(\varphi Y, \varphi W) - g(\varphi Y, \varphi Z) g(\varphi X, \varphi W) \right]. \]  

(3.13)

By virtue of (1.2), (2.3), (2.4) and (2.15), (3.13) yields

\[ 0 = f_i \left[ g(Y,Z) g(X,W) - g(Y,Z) \eta(X) \eta(W) - g(X,W) \eta(Y) \eta(Z) \right] \\
+ g(X,Z) g(Y,W) + g(Y,W) \eta(X) \eta(Z) + g(X,Z) \eta(Y) \eta(W) \\
+ f_i \left[ g(X,\varphi Z) g(\varphi Y,W) - g(\varphi Y,Z) g(X,\varphi W) \right] \\
+ 2 g(X,\varphi Y) g(\varphi Z,W) \right] + \frac{1}{2(n-1)} \left[ S(X,Z) g(Y,W) \right] \\
- S(X,Z) \eta(Y) \eta(W) - 2n(f_i - f_j) g(Y,W) \eta(X) \eta(Z) \\
- S(Y,Z) g(X,W) + S(Y,Z) \eta(X) \eta(W) + 2n(f_i - f_j) g(X,W) \eta(Y) \eta(Z) \\
+ S(Y,W) g(X,Z) - S(Y,W) \eta(X) \eta(Z) - 2n(f_i - f_j) g(X,Z) \eta(Y) \eta(W) \\
- S(X,W) g(Y,Z) + S(X,W) \eta(Y) \eta(Z) + 2n(f_i - f_j) g(Y,Z) \eta(X) \eta(W) \right] \\
- \frac{k-2}{2(n-1)} \left[ g(X,Z) g(Y,W) - g(X,Z) \eta(Y) \eta(W) - g(Y,W) \eta(X) \eta(Z) \right] \\
- g(Y,Z) g(X,W) + g(Y,Z) \eta(X) \eta(W) + g(X,W) \eta(Y) \eta(Z) \right]. \]  

(3.14)

Let \( \{ e_i : i = 1,2,\ldots,2n+1 \} \) be an orthonormal basis of the tangent space at any point of
the manifold. Putting $X = W = e_i$ in (3.14) and taking summation over $i$, $1 \leq i \leq 2n+1$, we get

$$0 = (2n-1)f_i\left\{g(Y, Z) - \eta(Y)\eta(Z)\right\}$$

$$+ f_i\left\{3g(\varphi Y, \varphi Z) - g(\varphi Y, Z)Tr \varphi\right\}$$

$$+ \frac{1}{2(n-1)}[-2(n-1)S(Y, Z) - S(Z, \xi)\eta(Y)]$$

(3.15)

$$- S(Y, \xi)\eta(Z) + \{2n(2n-1)(f_i - f_i) + r\}\eta(Y)\eta(Z)$$

$$+ \frac{r+2}{2(n-1)}[g(Y, Z) - \eta(Y)\eta(Z)].$$

By the use of (2.3) and (2.11), (3.15) reduces to

$$S(Y, Z) = \left[\frac{(2n^2 - 2n + 1)f_i + 3(n-1)f_2 - nf_3 + 1}{n-1}\right]g(Y, Z)$$

$$+ \left[\frac{n(3-2n)f_i - f_i - 3(n-1)f_2 - 1}{n-1}\right]\eta(Y)\eta(Z),$$

(3.16)

under the condition $Tr \varphi = 0$.

From (3.16) we get

$$S(Y, Z) = \alpha g(Y, Z) + \beta \eta(Y)\eta(Z).$$

(3.17)

where

$$\alpha = \frac{(2n^2 - 2n + 1)f_i + 3(n-1)f_2 - nf_3 + 1}{n-1}$$

and

$$\beta = \frac{n(3-2n)f_i - f_i - 3(n-1)f_2 - 1}{n-1}.$$

The relation (3.17) implies that the manifold is an $\eta$-Einstein manifold. This completes the proof of the theorem.

**Theorem 3.4.** A $(2n+1)$-dimensional generalized Sasakian-space-form $M(f_1, f_2, f_3)$ satisfying the condition $B(\xi, X)S = 0$ is an Einstein manifold and has the scalar curvature $r = 2n(2n+1)(f_i - f_i)$.

**Proof.** Let $M(f_1, f_2, f_3)$ be a $(2n+1)$-dimensional generalized Sasakian-space-form. Suppose that $M(f_1, f_2, f_3)$ satisfies the condition $(B(\xi, X)S)(U, V) = 0$, where $S$ is the Ricci tensor. Then we have

$$S(B(\xi, X)U, V) + S(U, B(\xi, X)V) = 0.$$
In view of (2.1), (2.2), (2.7) and (2.11), (2.16) yields

\[
B(\xi, Y)Z = \left( f_i - f_3 - \frac{1}{n-1} \right) \left\{ g(Y, Z)\xi - \eta(Z)Y \right\} \\
+ \frac{1}{2(n-1)} [2n(f_i - f_3)(Y - \eta(Y)\xi)\eta(Z)] \\
- \{g(Y, Z) - \eta(Y)\eta(Z)\}Q\xi].
\]

(3.19)

Using (3.19) in (3.18) we get

\[
0 = \left( f_i - f_3 - \frac{1}{n-1} \right) \left\{ 2n(f_i - f_3)g(X, U)\eta(V) - S(X, U)\eta(U) \right\} \\
+ \frac{n(f_i - f_3)}{n-1} [\{S(X, V) - 2n(f_i - f_3)\eta(X)\eta(V)\}\eta(U)] \\
- 2n(f_i - f_3)\{g(X, U) - \eta(X)\eta(U)\}\eta(V)] \\
+ \left( f_i - f_3 - \frac{1}{n-1} \right) \left\{ 2n(f_i - f_3)g(X, V)\eta(U) - S(X, U)\eta(V) \right\} \\
+ \frac{n(f_i - f_3)}{n-1} [\{S(X, U) - 2n(f_i - f_3)\eta(X)\eta(U)\}\eta(V)] \\
- 2n(f_i - f_3)\{g(X, V) - \eta(X)\eta(V)\}\eta(U)].
\]

(3.20)

Putting \(V = \xi\) in (3.20) and using (2.1), (2.2) and (2.11), we obtain

\[
S(X, U) = 2n(f_i - f_3)g(X, U).
\]

(3.21)

The relation (3.21) implies that the generalized Sasakian-space-form is an Einstein manifold.

Again, taking an orthonormal frame field at any point of the manifold and contracting over \(X\) and \(U\) in (3.21) we have

\[
r = 2n(2n + 1)(f_i - f_3),
\]

(3.22)

where \(r\) is the scalar curvature.

In view of (3.21) and (3.22), the theorem is proved.

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