ON THE 3rd ORDER LINEAR DIFFERENTIAL EQUATION

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ABSTRACT
If for an arbitrary 3th order linear differential equation, non-homogeneous, we know two solutions of its associated homogeneous equation (HE), then we show how to determine the third solution of HE and the particular solution of the original equation.

Keywords: Wronskian, Linear differential equations, Method of variation of parameters

INTRODUCTION
If for the linear differential equation of third order:

$$p(x)y'' + q(x)y' + r(x)y = \Phi(x)$$  \hfill (1)

we know the solution $y_1$ of the corresponding homogeneous equation (HE):

$$p y'' + q y' + r y = 0$$  \hfill (2)

then it is possible to obtain the solution $y_2$ of (2) and the particular solution $y_p$ of (1) [1-5]:

$$y_2(x) = y_1(x) \int_{\frac{x}{p}}^{\frac{x}{p}} \frac{p y}{W} \, d\eta, \quad y_p(x) = y_2(x) \int_{\frac{x}{p}}^{\frac{x}{p}} \frac{y_1 \phi}{W} \, d\eta - y_1(x) \int_{\frac{x}{p}}^{\frac{x}{p}} \frac{y_2 \phi}{W} \, d\eta,$$  \hfill (3)

where $W$ is the Wronskian of the two independent solutions of (2), with the Abel – Liouville – Ostrogradski identity:

$$W \equiv y_1 y_2' - y_2 y_1' = \exp \left(-\int_{\frac{x}{p}}^{\frac{x}{p}} \frac{q}{r} \, d\xi \right)$$  \hfill (4)
The expression (3) for $y_p$ can be constructed via method of variation of parameters of Euler (1741) – Lagrange (1777), or employing the technique of adjoint-exact linear differential operator[4,5].

Here we consider the differential equation of third order:

$$u(x)y'''+p(x)y''+q(x)y'+r(x)y = \phi(x),$$

and we accept the knowledge of the solutions $y_1$ & $y_2$ of its HE:

$$u(y'''+p y''+q y'+r y = 0),$$

with the aim to find expressions for the particular solution of (5) and the solution $y_3$ of (6).

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In this case, the HE (6) has three solutions:

$$u(y_j'''+p y_j''+q y_j'+r y_j = 0), \quad j = 1, 2, 3$$

whose linear independence implies a non-null Wronskian:

$$W = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix}.$$  \hspace{1cm} (8)

The derivative of (8) gives:

$$\frac{dW}{dx} = y_1'''+y_2'''+y_3'''+W_{23} y_1''+W_{31} y_2''+W_{12} y_3''.$$  \hspace{1cm} (9)

with the notation:

$$W_{i,j} = -W_{j,i} = y_i y_j'-y_j y_i', \quad i \neq j.$$  \hspace{1cm} (10)

If (9) is multiplied by $u(x)$ and we use (7), then:

$$u \frac{dW}{dx} = -p W \quad \therefore W = k \exp\left(-\int \frac{p}{u} d\xi\right).$$
but, without loss of generality, we may take k=1 because we can multiply the $y_j$ by an adequate scale factor (they are solutions of a HE), therefore:

$$W = \exp \left(-\int \frac{p}{u} \, d\xi \right),$$

(11)

is the Abel – Liouville – Ostrogradski identity for (5).

The expansion of the determinant (8), via the third column, implies:

$$W_{12} \, y_3'' - W_{12} \, y_3' + (y_1' \, y_2'' - y_2' \, y_1'') \, y_3 = W,$$

(12)

where, in accordance with (10):

$$W_{12} = y_1 \, y_2' - y_2 \, y_1' \quad \text{and} \quad W_{12} = \frac{d}{dx} \, W_{12} = y_1 \, y_2'' - y_2 \, y_1''.$$

(13)

It is interesting to see that $y_3$ satisfies the HE (6) of 3\textsuperscript{rd} order, and besides it is a particular solution of the non-homogeneous equation (12) of 2\textsuperscript{nd} order. It is simple to verify that $y_1$ & $y_2$ are solutions of the HE of (12):

$$W_{12} \, y_c'' - W_{12} \, y_c' + (y_1' \, y_2'' - y_2' \, y_1'') \, y_c = 0, \quad c = 1, 2,$$

(14)

then the method of variation of parameters gives the particular solution for (12):

$$y_3(x) = y_2(x) \int x \frac{y_3 \, W}{(W_{12})^2} \, d\eta - y_1(x) \int x \frac{y_3 \, W}{(W_{12})^2} \, d\eta,$$

(15)

thus $y_3$ is determined employing $y_1$ & $y_2$.

With (8) and (10) it is easy to prove the identities:

$$y_1 \, W_{23} + y_2 \, W_{31} + y_3 \, W_{12} = 0,$$

$$y_1' \, W_{23} + y_2' \, W_{31} + y_3' \, W_{12} = 0,$$

(16)

$$y_1'' \, W_{23} + y_2'' \, W_{31} + y_3'' \, W_{12} = W,$$
which permit to construct the particular solution of (5):

\[ y_p(x) = y_1(x) \int_x^\infty \frac{w_{3\lambda}}{w} \frac{\phi}{u} \, d\eta + y_2(x) \int_x^\infty \frac{w_{2\lambda}}{w} \frac{\phi}{u} \, d\eta + y_3(x) \int_x^\infty \frac{w_{1\lambda}}{w} \frac{\phi}{u} \, d\eta, \]  

(17)

with \( W \) given by (11).

The relations (15) and (17) are the generalizations of (3) for the 3\(^{rd}\) order case, and they are not explicitly given in the literature.

REFERENCES


