SOME MATRIX TRANSFORMATIONS AND ALMOST CONVERGENCE

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ABSTRACT
The sequence space $bv(u, p)$ has been defined and the classes $(bv(u, p): l_\infty)$, $(bv(u, p): c)$ and $(bv(u, p): c_0)$ of infinite matrices have been characterized by Başar, Altay and Mursaleen (see, [2]). The main purposes of the present paper is to characterize the classes$(bv(u, p): f_\alpha)$, $(bv(u, p): f)$ and$(bv(u, p): f_0)$, where $f_\alpha$, $f$ and $f_0$ denotes the spaces of almost bounded sequences, almost convergent sequences and almost convergent null sequences, respectively, with real or complex terms.

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1. INTRODUCTION, BACKGROUND AND PRELIMINARIES
A sequence space is defined to be a linear space with real or complex sequences. Throughout the paper $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{C}$ denotes the set of non-negative integers, the set of real numbers and the set of complex numbers, respectively. Let $\ell_\infty$, $c$ and $c_0$ respectively be Banach spaces of bounded, convergent and null sequences $x = \{x_n\}_{n=0}^{\infty}$ normed by $\|x\| = \sup_{n \geq 0} |x(n)|$; also, by $cs$ we denote the sequence of all convergent series.(see, [7]).

Let $X$ and $Y$ be two non-empty subsets of the space $\omega$ of real or complex sequences. Let $A = (a_{nk})(n,k \in \mathbb{N})$, be an infinite matrix of real or complex numbers. We write $(Ax)_n = A_n(x) = \sum_k a_{nk}x_k$. Then $Ax = \{A_n(x)\}$ is called the $A$-transform of $x$, whenever $A_n(x) = \sum_k a_{nk}x_k$ converges for each $n \in \mathbb{N}$. We write $\lim_n Ax = \lim_n A_n(x)$. If $x \in X$ implies $Ax \in Y$, we say that $A$ defines a (matrix) transformation from $X$ into $Y$ and we denote it by $A: X \rightarrow Y$. By $(X: Y)$, we mean the class of all matrices $A$ such that $A: X \rightarrow Y$.

Let $D$ denote the shift operator on $\omega$, that is, $Dx = \{x(n)\}_{n=1}^{\infty}$, $D^2x = \{x(n)\}_{n=2}^{\infty}$ and so on. Obviously, $D$ is a bounded linear operator on $l_\infty$ onto itself. A Banach limit $L$ is a non-negative linear functional on $l_\infty$ such that $L$ is invariant under the shift operator that is, $L(Sx) = L(x)$ and that $L(e) = 1$, where $e = \{1,1,...\}$ (see, [1]). A sequence space is said to be almost convergent (see, [3]) to the generalized limit $\alpha$ if all Banach limits of $x$ are $\alpha$. We denote the set of almost convergent sequences by $f$. It was proved by Lorentz (see, [3]) that

\[
f = \{ x \in l_\infty : \lim_m \tau_{mn}(x) = \alpha , uniformly in n \},
\]

where, \[
\tau_{mn}(x) = \frac{1}{m+1} \sum_{j=0}^{m} x_{j+n} , \tau_{-1,n} = 0 \text{ and } \alpha = f-\lim x.
\]

Nanda [6] has defined a new set of sequences $f_\alpha$, as follows:

\[
f_\alpha = \{ x \in l_\infty : \lim_m |\tau_{mn}(x)| < \infty \}.
\]
We call $f_{\infty}$ the set of all almost bounded sequences. We denote by $X^\beta$, the $\beta$-dual of a sequence space $X$ and mean the set of all these sequences $x = (x_k)$ such that $xy = (x_k y_k) \in cs$ for all $y = (y_k) \in X$.

The approach of constructing a new sequence space by means of matrix domain of a particular limitation method has been studied by several authors viz., ([2, 4, 5]). The sequence space $bv(u, p)$ has been defined and the various classes ($bv(u, p)$: $l_{\infty}$) ($bv(u, p)$: $c$) and ($bv(u, p)$: $c_0$) have been characterized (see, [2]). In the present paper, we characterize the classes ($bv(u, p)$: $f_{\infty}$), ($bv(u, p)$: $f$) and ($bv(u, p)$: $f_0$), where $u = (u_k)$ is a sequence such that $u_k \neq 0$ for all $k \in \mathbb{N}$.

The space $bv(u, p)$ is defined (see, [2]) as $bv(u, p) = \{ x = (x_k) \in \omega : \sum_k |u_k \Delta x_k|^p < \infty \}$, where, $\Delta x_k = x_k - \Delta x_{k-1}$.

2. MAIN RESULTS
Define the sequence $y = (y_k)$ which will be used as the $A^u$-transform of a sequence $x = (x_k)$, i.e.,

$$y_k = u_k \Delta x_k ; \ k \in \mathbb{N}. \tag{2.1}$$

For brevity in notation, we write $t_{mn} (x) = \frac{1}{m+1} \sum_{j=0}^{m} A_{n+j} (x) = \sum_k a(n, k, m) x_k$.

where, $a(n, k, m) = \frac{1}{m+1} \sum_{j=0}^{m} a_{n+j, k} ; \ (n, k, m \in \mathbb{N})$

Also, $\bar{a}(n, k, m) = \left[ \frac{a(n, k, m)}{u_k} \right] ; (n, k, m \in \mathbb{N})$.

Now, we give the following lemmas which will be needed in proving the main Theorems.

**Lemma 2.1 [2]** Define the sets $D_1(p)$ and $D_2(p)$ as follows:

$D_1(p) = \left\{ a = (a_k) \in \omega : \sup_n \sum_k \left| \sum_{j=k}^{a_j} u_k \right|^p \right\}$

$D_2(p) = \bigcup_{B > 1} \left\{ a = (a_k) \in \omega : \sup_n \sum_k \left| \sum_{j=k}^{a_j} u_k ^{-1} B^{-1} \right|^p \right\}$

Then, $[bv(u, p)^\beta] = D_1(p) \cap cs ; \ (0 < p_k \leq 1)$ and $[bv(u, p)^\beta] = D_2(p) \cap cs ; \ (1 < p_k < \infty)$.

**Lemma 2.2 [6]** $f \subset f_{\infty}$.

We consider only the case $1 < p_k \leq M < \infty$ and the case $0 < p_k \leq 1$ may be proved in a similar fashion.

**Theorem 2.3**: (a) Let $1 < p_k \leq M < \infty$ for every $k \in \mathbb{N}$. Then $A \in (bv(u, p): f_{\infty})$ if and only if

$$\sup_{n, m} \sum_k |\bar{a}(n, k, m) B^{-1}|^{p^k} < \infty \tag{2.2}$$

and $\{a_{nk}\} \in D_2(p) \cap cs$. \tag{2.3}
(b) Let $0 < p_k \leq 1$ for every $k \in \mathbb{N}$. Then $A \in (bv(u,p); f_\infty)$ if and only if
\[ \sup_{n,m} \sum_k |\bar{a}(n,k,m)|^p_k < \infty \]  
and
\[ \{a_{nk}\} \in D_1(p) \cap cs. \]  

Proof: Sufficiency: Suppose the conditions hold and $x \in bv(u,p)$. Using the inequality which holds for any $C > 0$ and any two complex numbers $a, b$
\[ |ab| \leq C(|a|^{q} + |b|^p), \]
where, $p > 1$ and $p^{-1} + q^{-1} = 1$ (see, [3]), we have
\[ |t_{mn}(Ax)| = |\sum_k a(n,k,m)x_k| = |\sum_k \bar{a}(n,k,m)y_k| \leq \sum_k B \left[ |\bar{a}(n,k,m)B^{-1}|^p_k + |y_k|^p_k \right] \]

Now, taking \( \sup \) over $m, n$ on both sides to the above inequality, we get $Ax \in f_\infty$ for every $x \in bv(u,p)$, i.e., $A \in (bv(u,p); f_\infty)$.

Necessity: Suppose that $A \in (bv(u,p); f_\infty)$. Then $Ax$ exists for every $x \in bv(u,p)$, and this implies that $\{a_{nk}\} \in [bv(u,p)]^\beta$ for every $n \in \mathbb{N}$, the necessity of (2.3) is immediate.

Now, $\sum_k a(n,k,m)x_k$ exists for each $m, n$ and $x \in bv(u,p)$, the sequences $\{a(n,k,m)\}_{k \in \mathbb{N}}$ define the continuous linear functionals $\varphi_{mn}(x)$ on $bv(u,p)$ by $\varphi_{mn}(x) = \sum_k a(n,k,m)x_k$; $n, k, m \in \mathbb{N}$. Since $bv(u,p)$ is complete and $\sup_{m,n}|\sum_k \bar{a}(n,k,m)x_k| < \infty$, so by uniform bounded principle, there exists $M > 0$ such that
\[ \sup_{m,n} |\varphi_{mn}(x)| = \sup_{m,n} |\sum_k a(n,k,m)x_k| = \sup_{m,n} |\sum_k \bar{a}(n,k,m)x_k| \leq M < \infty. \]

This implies that $\sup_{m,n} |\sum_k \bar{a}(n,k,m)x_k|^p_k < \infty$, which shows the necessity of the condition (2.2) and the proof of (i) is complete.

Theorem 2.4: (a) Let $1 < p_k \leq M < \infty$ for every $k \in \mathbb{N}$. Then $A \in (bv(u,p); f_\infty)$ if and only if
(i) the condition (2.2)-(2.5) of Theorem 2.3 holds
(ii) there is a sequence $(\beta_k)$ of scalars such that
\[ \lim_m \bar{a}(n,k,m) = \beta_k, \quad \text{uniformly in } n. \]  

Proof: Sufficiency: Suppose that the conditions (2.2)-(2.6) hold and $x \in bv(u,p)$. Then $Ax$ exists and we have by (2.6) that $|\bar{a}(n,k,m)B^{-1}|^p_k \to |\beta_k B^{-1}|^p_k$ as $m \to \infty$ uniformly in $n$ for each $k \in \mathbb{N}$, which leads us with (2.2) that
\[ \sum_{j=0}^{k} |\beta_j B^{-1}|^p_k = \sum_{j=0}^{k} |\bar{a}(n,j,m)B^{-1}|^p_k \leq \sup_{m,n} \sum_j |\bar{a}(n,j,m)B^{-1}|^p_k < \infty, \]
holding for every $k \in \mathbb{N}$. Consequently reasoning as in the proof of the sufficiency of Theorem 2.3, the series $\sum_k a(n,k,m)x_k$ and $\sum_k \beta_kx_k$ converges for every $n, m$ and for every $x \in bv(u,p)$. Now, for given $\epsilon > 0$ and $x \in bv(u,p)$, choose a fixed $k_0 \in \mathbb{N}$ such that
\[\left[\sum_{k=k_0+1}^{\infty}|x_k|^p\right]^{\frac{1}{p}} < \varepsilon, \text{ where } H = \sup_{k} p_k. \] Then, there is some \(m_0 \in \mathbb{N}\), by condition (ii) such that \[\left|\sum_{k=1}^{k_0} [a(n,k,m) - \beta_k]\right| < \varepsilon, \text{ for every } m \geq m_0 \text{ and uniformly in } n. \]

Now, since \(\sum_k a(n,k,m)x_k\) and \(\sum_k \beta_k x_k\) converges (absolutely) uniformly in \(n,m\) and for \(x \in bv(u,p)\), we have that \(\sum_{k=0}^{k_0}[a(n,k,m) - \beta_k]x_k < \frac{\varepsilon}{2}\), converges uniformly in \(n,m\) and \(x \in bv(u,p)\). Hence by conditions (i) and (ii) we have \(\sum_{k=0}^{\infty}[a(n,k,m) - \beta_k] \rightarrow 0\) (for all \(m \geq m_0\)) uniformly in \(n\). Therefore, \(\sum_{k=0}^{\infty}[a(n,k,m) - \beta_k] \rightarrow 0\) uniformly in \(i.e.,\)

\[
\lim_m \sum_k a(n,k,m)x_k = \sum_k \beta_k x_k \text{ uniformly in } n.
\]

Hence, \(Ax \in f\), which proves sufficiency.

**Necessity:** Suppose that \(A \in (bv(u,p):f)\). Then, since \(f \subset f_\infty\) (by Lemma 2.1), the necessities of condition (i) is immediately obtained from Theorem 2.1. To prove the necessity of (ii) \(i.e.,(2.6)\), consider the sequence \(e_k = (0,0,...,1^{k\text{th place}},0,0,...) \in bv(u,p)\), condition (ii) follows immediately by (2.7) and the proof is complete.

**Collary 2.5:** \(A \in (bv(u,p):f_0)\) if and only if condition (i) and (ii) of above Theorem holds along with \(\beta_k = 0\) for each \(k \in \mathbb{N}\).

**Proof:** The proof follows from theorem 2.4 by taking \(\beta_k = 0\) for each \(k \in \mathbb{N}\).

**REFERENCES**


