COMMON FIXED POINT OF SEMI COMPATIBLE MAPS IN FUZZY METRIC SPACES

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ABSTRACT
The purpose of this paper is to prove a common fixed point theorem on fuzzy metric space using the notion of semi compatibility, our result generalize the result of Som [8]. Also, we are giving an example that make strong to our result.

Keywords : Common fixed point, Fuzzy metric space, R- weakly commuting , Semi compatible maps.

AMS Subject Classification : 47H10, 54H25.

INTRODUCTION
It proved a turning point in the development of mathematics when the notion of fuzzy set was introduced by Zadeh [10], which laid the foundation of fuzzy mathematics. Kramosil and Michalek [4] introduced the concept of fuzzy metric space and modified by George and Veeramani [2]. Also Grabiec [3] has proved some fixed point results for fuzzy metric space. Sessa [6] proved some theorems of commutativity by weakening the condition to weakly commutativity. Vasuki [9] defined the R- weak commutativity of mappings of Fuzzy metric space and proved the fuzzy version of Pant’s [5] theorem. Cho, Sharma and Sahu [1] introduced the concept of semi compatibility of mapps in D- metric space if condition (a) Sy = Ty implies that STy = TSy and (b) {Txₙ}→x, {Sxₙ}→x then {STxₙ}→Tx as n→∞ hold. However (b) implies (a) taking {xₙ}→y and x = Ty = Sy. So, here we define semi compatibility by condition (b) only. In this paper we used the concept of semi compatible mappings to prove further results.

PRELIMINARIES AND DEFINITIONS
Definitions 2.1.[7] *: [0,1] × [0,1] → [0,1] is a continuous t- norm if it satisfies the following conditions:
(i) * is associative and commutative,
(ii) * is continuous,
(iii) a * 1 = a ∀ a ∈ [0,1]
(iv) a * b ≤ c * d whenever a ≤ c and b ≤ d, for each a, b, c, d ∈ [0,1].
Definition 2.2. [4] The triplet \((X, M, \ast)\) is said to be Fuzzy metric space if \(X\) is an arbitrary set, \(\ast\) is a continuous t-norm and \(M\) is a Fuzzy set on \(X \times X \times [0, \infty] \rightarrow [0,1]\) satisfying the following conditions:

(FM-1) \(M(x, y, 0) = 0\),

(FM-2) \(M(x, y, t) = 1\) for all \(t > 0\) if and only if \(x = y\),

(FM-3) \(M(x, y, t) = M(y, x, t)\),

(FM-4) \(M(x, y, t) \ast M(y, z, s) \leq M(x, z, t+s)\),

(FM-5) \(M(x, y, \cdot) : [0, \infty) \rightarrow [0,1]\) is left continuous,

(FM-6) \(\lim_{t \rightarrow \infty} M(x, y, t) = 1\).

Note that \(M(x, y, t)\) can be considered as the degree of nearness between \(x\) and \(y\) with respect to \(t\). We identify \(x = y\) with \(M(x, y, t) = 1\) for all \(t > 0\). The following example shows that every metric space induces a Fuzzy metric space.

Example 2.1. [2] Let \((X, d)\) be a metric space. Define \(\ast = \min\{a, b\}\) and

\[M(x, y, t) = \frac{t}{t + d(x, y)}\]

for all \(x, y \in X\) and all \(t > 0\). Then \((X, M, \ast)\) is a Fuzzy metric space. It is called the Fuzzy metric space induced by \(d\).

Lemma 2.1. [3] For all \(x, y \in X\), \(M(x, y, \cdot)\) is a non-decreasing function.

Definition 2.3. [3] A sequence \(\{x_n\}\) in a Fuzzy metric space \((X, M, \ast)\) is said to be a Cauchy sequence if and only if for each \(\varepsilon > 0, t > 0\), there exists \(n_0 \in \mathbb{N}\) such that \(M(x_n, x_m, t) > 1 - \varepsilon\) for all \(n, m \geq n_0\).

The sequence \(\{x_n\}\) is said to converge to a point \(x\) in \(X\) if and only if for each \(\varepsilon > 0, t > 0\), there exists \(n_0 \in \mathbb{N}\) such that \(M(x_n, x, t) > 1 - \varepsilon\) for all \(n \geq n_0\).

A Fuzzy metric space \((X, M, \ast)\) is said to be complete if every Cauchy sequence in it converge to a point in it.

Definition 2.4. [5] Two self maps \(A\) and \(S\) of Fuzzy metric space \((X, M, \ast)\) are said to be weakly commuting if

\[M(ASx, SAx, t) \geq M(Ax, Sx, t)\]

for all \(x \in X\). The notion of weak commutativity is extended to \(R\)-weak commutativity by Vasuki [9] as

Definition 2.5. [9] Two self maps \(A\) and \(S\) of Fuzzy metric space \((X, M, \ast)\) are said to be \(R\)-weakly commuting provided there exist some positive real number \(R\) such that

\[M(ASx, SAx, t) \geq M(Ax, Sx, \frac{t}{R})\]

for all \(x \in X\). The weak commutativity implies \(R\)-weak commutativity and converse is true for \(R \leq 1\).

Definition 2.6. A pair \((A, S)\) of self mappings of a Fuzzy metric space is said to be Semi compatible if \(M(ASx_n, Sx_n, t) \rightarrow 1\) for all \(t > 0\) whenever \(\{x_n\}\) is a sequence in \(X\) such that \(Ax_n, Sx_n \rightarrow p\) for some \(p\) in \(X\) as \(n \rightarrow \infty\).

It follows that \((A, S)\) is Semi compatible and \(Ay = Sx\) imply \(ASy = SAy\) by taking \(\{x_n\} = y\) and \(x = Ay = Sy\).

Remark 2.1. Let \((A, S)\) be a pair of self mappings of a Fuzzy metric space \((X, M, \ast)\). Then \((A, S)\) is \(R\)-weakly commuting implies \((A, S)\) is Semi compatible but the converse is not true.
Using R-weak commutativity, Som [8] proved some results. Here we generalized the result of Som [8] by replacing the assumption of R-weakly commuting maps to Semi compatible maps.

**Example 2.2.** Let \( X = [0, 2] \) and \( a * b = \min \{a, b\} \). Let \( M(x, y, t) = \frac{t}{t + d(x, y)} \) be the standard Fuzzy metric space induced by \( d \), where \( d(x, y) = |x - y| \) for all \( x, y \in X \), define

\[
A(x) = \begin{cases} 
2, & x \in [0,1] \\
\frac{x}{2}, & x \in (1,2]
\end{cases} 
\]

\[
S(x) = \begin{cases} 
1, & x \in [0,1) \\
2, & x = 1 \\
\frac{x+3}{5}, & x \in (1,2]
\end{cases}
\]

Now for \( 1 < x \leq 2 \) we have

\[
Ax = \frac{x}{2}, \quad Sx = \frac{x+3}{5} \quad \text{and} \quad ASx = \frac{x+3}{10}, \quad SAx = \frac{x+6}{10}
\]

then

\[
M(ASx, SAx, t) = \frac{10t}{10t + 3} \quad \text{and} \quad M(Ax, Sx, \frac{t}{R}) = \frac{10t}{10t + 3(2-x)R}.
\]

We observe that \( M(ASx, SAx, t) \geq M(Ax, Sx, \frac{t}{R}) \) which gives \( R \geq \frac{1}{2-x} \)

Therefore we get there no \( R \) for \( x \in (1, 2] \) in \( X \).

Hence \( (A,S) \) is not R-weakly commuting.

Now we have \( S(1) = 2 = A(1) \), and \( S(2) = 1 = A(2) \)

also \( SA(1) = AS(1) \) and \( AS(2) = 2 = AS(2) \)

Let \( x_n = 2 - \frac{1}{2n} \)

Hence \( Ax_n \rightarrow 1, \quad Sx_n \rightarrow 1 \) and \( ASx_n \rightarrow 2 \)

Therefore \( M(ASx_n, Sy, t) = (2, 2, t) = 1 \).

Hence \( (A, S) \) is Semi compatible but not R-weakly commuting.

**MAIN RESULTS**

**Theorem 3.1.** Let \( S \) and \( T \) be two continuous self mappings of a complete Fuzzy metric space \( (X, M, *) \) such that \( a * b = \min \{a, b\} \) for all \( a, b \) in \( X \). Let \( A \) be a self mapping of \( X \) satisfying the following conditions:

1. \( A(X) \subseteq S(X) \cap T(X) \),
2. \( (A,S) \) and \( (A,T) \) are semi compatible,
3. \( M(Ax, Ay, t) \geq r \min\{M(Sx, Ty, t), M(Sx, Ax, t), M(Sx, Ay, t), M(Ty, Ay, t)\} \) for all \( x, y \in X \) and \( t > 0 \), where \( r : [0, 1] \rightarrow [0, 1] \) is a continuous function such that \( r(t) > t \), for each \( 0 < t < 1 \).

Then \( A, S, T \) have a unique common fixed point in \( X \).

**Proof:** Let \( x_0 \in X \) be any arbitrary point.

Since \( A(X) \subseteq S(X) \) then there must exists a point \( x_1 \in X \) such that \( Ax_0 = Sx_1 \).

Also, since \( A(X) \subseteq T(X) \), there exists another point \( x_2 \in X \) such that \( Ax_1 = Tx_2 \).
In general, we get a sequence \( \{y_n\} \) recursively as
\[
y_{2n} = Sx_{2n+1} = A x_{2n} \quad \text{and} \quad y_{2n+1} = T x_{2n+2} = A x_{2n+1}, \quad n \in \mathbb{N} \cup \{0\}.
\]

Let \( M_{2n} = M (y_{2n+1}, y_{2n}, t) = M (A x_{2n+1}, A x_{2n}, t) \). Then, \( M (A x_{2n+2}, A x_{2n+1}, t) = M_{2n+1} \).

Using inequality (3), we get
\[
M_{2n+1} \geq r \min \{ M(Sx_{2n+2}, T x_{2n+1}, t), M(Sx_{2n+2}, A x_{2n+2}, t), M(Sx_{2n+2}, A x_{2n+1}, t),
M(T x_{2n+1}, A x_{2n+1}, t) \}
= r \min \{ M(A x_{2n+1}, A x_{2n}, t), M(A x_{2n+1}, A x_{2n+2}, t), M(A x_{2n+1}, A x_{2n+1}, t),
M(A x_{2n}, A x_{2n+1}, t) \}
= r \min(M_{2n}, M_{2n+1}, M_{2n}) \tag{3.1}
\]

If \( M_{2n} > M_{2n+1} \), then by definition of \( r \) we have
\[
M_{2n+1} \geq r(M_{2n+1}) > M_{2n+1}, \quad \text{a contradiction. So, } M_{2n+1} \geq M_{2n}.
\]

Thus, from (3.1), we get \( M_{2n+1} \geq r(M_{2n}) \geq M_{2n} \) \quad \tag{3.2}

Hence \( \{M_{2n}\} \) where \( 0 \leq n \leq \infty \) is an increasing sequence of positive numbers in \([0, 1]\) and therefore, tends to a limit \( L \leq 1 \).

We claim that \( L = 1 \). If \( L < 1 \), then on taking limit \( n \to \infty \) in (3.2), we get
\[
L \geq r(L) \geq L;
\]
i.e. \( r(L) = L \), which contradicts the fact that \( L < 1 \).

Hence, \( L = 1 \).

Now for any positive integer \( p \),
\[
M(A x_n, A x_{n+p}, t) \geq M(A x_n, A x_{n+1}, \frac{t}{p}) \times M(A x_{n+1}, A x_{n+2}, \frac{t}{p}) \times \cdots \times M(A x_{n+p-1}, A x_{n+p}, \frac{t}{p})
> (1 - \varepsilon) \times (1 - \varepsilon) \times \cdots \times (1 - \varepsilon) \text{ (p-times) } = 1 - \varepsilon.
\]

Thus, \( M(A x_n, A x_{n+p}, t) > 1 - \varepsilon, \forall \ t > 0. \)
Hence $\{Ax_n\}$ is a Cauchy sequence in $X$. Since $X$ is complete $\{Ax_n\} \rightarrow z \in X$. Hence the subsequences $\{Sx_n\}$ and $\{Tx_n\}$ of $\{Ax_n\}$ also tends to the same limit.

**Case I.** Since $S$ is continuous. In this case we have

$SAx_n \rightarrow Sz, \quad SSx_n \rightarrow Sz$

Also $(A, S)$ is semi compatible, we have $ASx_n \rightarrow Sz$

**Step I.** Let $x = Sx_n, y = x_n$ in (3) we get

$$M(ASx_n, Ax_n, t) \geq r \min\{M(SSx_n, Tx_n, t), M(SSx_n, Ax_n, t), M(SSx_n, Ax_n, t), \ldots\}.$$  

Taking limit as $n \rightarrow \infty,$

$$M(Sz, z, t) \geq r \min\{M(Sz, z, t), M(Sz, Sz, t), M(Sz, z, t), M(z, z, t)\}.$$  

$$\geq r M(Sz, z, t),$$  

$$> M(Sz, z, t).$$

So, we get $Sz = z.$

**Step II.** By putting $x = z, y = x_n$ we get $Az = z.$

Hence, $Az = z = Sz.$

**Case II.** Since $T$ is continuous. In this case we have $TTx_n \rightarrow Tz, \quad TAx_n \rightarrow Tz.$

also $(A, T)$ is semi compatible $ATx_n \rightarrow Tz.$

**Step I.** Let $x = x_n, y = Tx_n$ in (3) we get

$$M(Ax_n, ATx_n, t) \geq r \min\{M(Sx_n, TTx_n, t), M(Sx_n, Ax_n, t), M(Sx_n, ATx_n, t), \ldots\}.$$  

$$M(TTx_n, ATx_n, t) \geq r \min\{M(z, Tz, t), M(z, z, t), M(z, Tz, t), M(Tz, Tz, t)\}.$$  

$$\geq r M(z, Tz, t),$$  

$$> M(z, Tz, t).$$
So, we get $Tz = z$. Thus, we have $Az = Sz = Tz = z$.

Hence $z$ is a common fixed point of $A$, $S$ and $T$.

**Uniqueness**: Let $u$ be another common fixed point of $A$, $S$ and $T$, Then

$$Au = Su = Tu = u.$$ 

Put $x = z$, $y = u$ in (3), we get

$$M(Az, Au, t) \geq r \min\{M(Sz, Tu, t), M(Sz, Az, t), M(Sz, Au, t), M(Tu, Au, t)\}.$$ 

Therefore

$$M(z, u, t) \geq r \min\{M(z, u, t), M(z, z, t), M(z, u, t), M(u, u, t)\}.$$ 

$$\geq r M(z, u, t),$$

$$> M(z, u, t)$$

which gives $z = u$.

Therefore $z$ is a unique common fixed point of $A$, $S$ and $T$.

If we take $T = S$ then we get following corollary

**Corollary 3.2.** let $S$ be a continuous mapping of a complete Fuzzy metric space $(X, M, *)$ such that $a * b = \min (a, b)$ for all $a, b$ in $X$. Let $A$ be a self mapping of $X$ satisfying the following conditions:

1. $A(X) \subseteq S(X)$,
2. $(A, S)$ is semi compatible,
3. $M(Ax, Sy, t) \geq r \min\{M(Sx, Sy, t), M(Sx, Ax, t), M(Sx, Ay, t), M(Sy, Ay, t)\}$ for all $x, y \in X$ and $t > 0$, where $r : [0, 1] \rightarrow [0, 1]$ is a continuous function such that
4. $r(t) > t$, for each $0 < t < 1$.

Then $A$ and $S$ have a common fixed point in $X$.

**Theorem 3.2.** Let $S$ and $T$ be two continuous self mappings of a complete Fuzzy metric space $(X, M, *)$ such that $a * b = \min (a, b)$ for all $a, b$ in $X$. Let $A$ and $B$ be two self mappings of $X$ satisfying the following conditions:

1. $A(X) \cup B(X) \subseteq S(X) \cap T(X)$,
2. $(A, T)$ and $(B, S)$ are semi compatible pairs,
3. $aM(Tx, Sy, t) + bM(Tx, Ax, t) + c M(Sy, By, t) + \max\{M(Ax, Sy, t), M(By, Tx, t)\} \leq q M(Ax, By, t)$
for all \( x, y \in X \), where \( a, b, c \geq 0 \) with \( q < (a + b + c) < 1 \).

Then \( A, B, S \) and \( T \) have a unique common fixed point in \( X \).

**Proof:** Let \( x_0 \in X \) be any arbitrary point.

Since \( A(X) \subset S(X) \) then there must exists a point \( x_1 \in X \) such that \( Ax_0 = Sx_1 \).

Also since \( A(X) \subset T(X) \), there exists another point \( x_2 \in X \) such that \( Ax_1 = Tx_2 \).

In general, we get a sequence \( \{y_n\} \) recursively as

\[
y_{2n} = Sx_{2n+1} = Ax_{2n} \quad \text{and} \quad y_{2n+1} = Tx_{2n+2} = Ax_{2n+1}, \quad n \in \mathbb{N} \cup \{0\}.
\]

Using inequality (3), we get similarly as som [9] that for \( \frac{a + b}{q - c} > 1 \) a Cauchy sequence in \( X \).

Hence, the sequence \( \{Ax_{2n}\}, \{Bx_{2n+1}\}, \{Sx_{2n+1}\} \) and \( \{Tx_{2n+2}\} \) are Cauchy and converge to same limit, say \( z \).

**Case I.** Since \( T \) is continuous. In this case we have

\[
TAx_n \to Tz, \quad TTx_n \to Tz
\]

Also \( (A, T) \) is semi compatible, we have \( ATx_n \to Tz \)

**Step I.** Let \( x = Tx_n, y = x_n \) in (3), we get

\[
aM(TTx_n, Sx_n, t) + bM(Tx_n, ATx_n, t) + cM(Sx_n, Bx_n, t)
+ \max\{M(ATx_n, Sx_n, t), M(Bx_n, TTx_n, t)\} \leq qM(ATx_n, Bx_n, t)
\]

Taking limit as \( n \to \infty \), we get

\[
aM(Tz, z, t) + bM(z, Tz, t) + cM(z, z, t)
+ \max\{M(Tz, z, t), M(z, Tz, t)\} \leq qM(Tz, z, t)
\]

i.e., \( aM(Tz, z, t) + bM(z, Tz, t) + c + M(Tz, z, t) \leq qM(Tz, z, t) \)

i.e., \( c \leq (q - a - b - 1) M(Tz, z, t) \)

i.e., \( M(Tz, z) \geq \frac{c}{q - a - b - 1} > 1 \)
which gives $Tz = z$.

**Step II.** Putting $x = z$ and $y = x_n$ in (3) we get

$$aM(Tz, Sx_n, t) + bM(Tz, Az, t) + cM(Sx_n, Bx_n, t)$$

$$+ \max\{M(Az, Sx_n, t), M(Bx_n, Tz, t)\} \leq qM(Az, Bx_n, t)$$

Taking limit as $n \to \infty$, we get

$$aM(z, z, t) + bM(z, Az, t) + cM(z, z, t)$$

$$+ \max\{M(Az, z, t), M(z, z, t)\} \leq qM(Az, z, t)$$

i.e. $a + bM(z, Az, t) + c + \max\{M(Az, z, t), 1\} \leq qM(Az, z, t)$

i.e. $a + c + 1 \leq (q - b) M(Az, z, t)$

i.e $M(Az, z, t) \geq \frac{a + c + 1}{q - b} > 1$

which gives $Az = z$.

Hence, $Az = z = Tz$.

**Case II.** Similarly since $S$ is continuous and $(B, S)$ is semi compatible

we get $Bz = z = Sz$.

Thus we have $Az = Bz = Tz = Sz = z$.

Hence $z$ is a common fixed point of $A$, $B$, $S$ and $T$, and easily we can prove that it is a unique common fixed point of $A$, $B$, $S$ and $T$.

**REFERENCES**


